Math 005C Exam 3

1. (6 pts) Evaluate.

$$\int_0^1 \int_0^2 y e^{x-y}\, dx\, dy = \int_0^1 y e^{-y} dy \int_0^2 e^x dx$$

$$u = y \qquad dv = e^{-y} dy$$

$$du = dy \qquad v = -e^{-y}$$

$$= \left(-y e^{-y} - e^{-y}\right|_0^1 \; e^x \Big|_0^2$$

$$= \left(-e^{-1} - e^{-1} + 1\right)\left(e^2 - 1\right)$$

$$= \left(1 - 2e^{-1}\right)\left(e^2 - 1\right)$$

$$= e^2 - 2e - 1 + 2e^{-1}$$

OR $$\int_0^1 \int_0^2 y e^{x-y}\, dx\, dy = \int_0^1 \left(y e^{2-y} - y e^{-y}\right) dy$$

$$= \left(e^2 - 1\right) \int_0^1 y e^{-y} dy = \left(e^2 - 1\right)\left(-y e^{-y} - e^{-y}\right|_0^1$$

$$= \left(e^2 - 1\right)\left(-e^{-y} - e^{-y} + 1\right) = \left(e^2 - 1\right)\left(1 - 2e^{-1}\right)$$

2. (5 pts) A lamina occupies the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant. Using an appropriate coordinate system for the lamina, set up the integrals you would use to find $\bar{x}$ (the x-coordinate of the center of mass) if the density at any point is proportional to its distance from the x-axis. Do not evaluate the integrals.

$$\rho(x,y) = ky \Rightarrow \rho(r,\theta) = kr\sin\theta$$

$$m = \int_0^{\pi/2}\int_0^1 kr\sin\theta \, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 kr^2\sin\theta\,d\theta$$

$$M_y = \int_0^{\pi/2}\int_0^1 kr\sin\theta(r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 kr^3\sin\theta\cos\theta\,dr\,d\theta$$

$$\bar{X} = \frac{M_y}{m}$$

3. (9 pts) Evaluate $\iint_D (x^2 + 2y)\, dA$, where D is bounded by $y = x$, $y = x^3$, $x \geq 0$. Set up integrals using both orders of integration ($dx\, dy$ and $dy\, dx$) and then compute either integral.

$$\int_0^1 \int_y^{\sqrt[3]{y}} (x^2+2y)\, dx\, dy = \int_0^1 \int_{x^3}^{x} (x^2+2y)\, dy\, dx$$

$$= \int_0^1 \left(x^2 y + y^2 \Big|_{x^3}^{x} \right) dx$$

$$= \int_0^1 (x^3 + x^2 - x^5 - x^6)\, dx$$

$$= \frac{23}{84}$$

4. (11 pts total) Set up integrals as specified that give the volume enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$.

a) (3 pts) Triple integral, cylindrical coordinates.

$$r \le z \le 2, \quad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi$$

$$V = \int_0^{2\pi} \int_0^2 \int_r^2 r\,dz\,dr\,d\theta$$

b) (4 pts) Double integral, rectangular coordinates with the order $dy\,dx$.

$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \quad -2 \le x \le 2$$

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(2 - \sqrt{x^2+y^2}\right) dy\,dx$$

c) (4 pts) Now actually compute the volume.

From (a): $$V = \int_0^{2\pi} \int_0^2 \int_r^2 r\,dz\,dr\,d\theta$$

$$= 2\pi \int_0^2 (2r - r^2)\,dr = 2\pi \left(r^2 - \tfrac{1}{3}r^3\right)\Big|_0^2$$

$$= \frac{8\pi}{3}$$

5. (9 pts) Evaluate $\iiint_D e^{-(x^2+y^2+z^2)^{3/2}}\, dV$, given that D is the portion of the sphere $x^2+y^2+z^2=4$ above Quadrants I and II of the xy-plane.

Spherical: $0 \le \varphi \le \pi/2$, $0 \le \theta \le \pi$, $0 \le \rho \le 2$

$$\int_0^{\pi/2}\int_0^{\pi}\int_0^{2} e^{-\rho^3}\rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= \int_0^{\pi/2} \sin\varphi\, d\varphi \int_0^{\pi} d\theta \int_0^{2} e^{-\rho^3}\rho^2\, d\rho$$

$$= 1(\pi)\left(-\tfrac{1}{3}e^{-\rho^3}\Big|_0^2\right.$$

$$= -\frac{\pi}{3e^8} + \frac{\pi}{3} = \underline{\frac{\pi}{3}\left(1-e^{-8}\right)}$$

6. (12 pts) Use the transformation $u = x - 3y, v = 3x + y$ to find

$$\iint_R \frac{x-3y}{(3x+y)^2}\, dA$$

where R is the rectangular region enclosed by the lines $x - 3y = 0, x - 3y = 4, 3x + y = 1$, and $3x + y = 3$.

$$0 \le u \le 4 \qquad 1 \le v \le 3$$

$$\begin{array}{rl} 3u = & 3x - 9y \\ -v = & -3x - y \\ \hline 3u - v = & -10y \end{array} \qquad \begin{array}{rl} u = & x - 3y \\ 3v = & 9x + 3y \\ \hline u + 3v = & 10x \end{array}$$

$$y = \frac{1}{10}(v - 3u) \qquad x = \frac{1}{10}(u + 3v)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{10} & \frac{3}{10} \\ -\frac{3}{10} & \frac{1}{10} \end{vmatrix} = \frac{1}{100} + \frac{9}{100} = \frac{1}{10}$$

$$\int_1^3 \int_0^4 \frac{u}{v^2} \cdot \frac{1}{10}\, du\, dv = \frac{1}{10} \int_1^3 v^{-2}\, dv \int_0^4 u\, du$$

$$= \frac{1}{10}\left(-\frac{1}{v}\right|_1^3 \frac{1}{2}u^2\Big|_0^4 = \frac{8}{15}$$